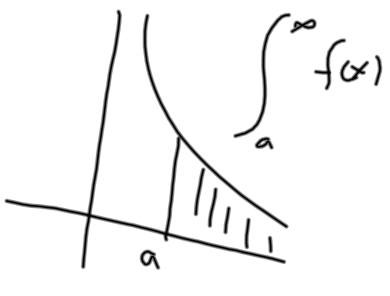
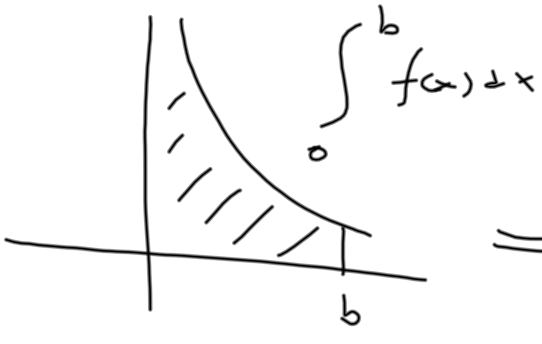
Improper integrals:

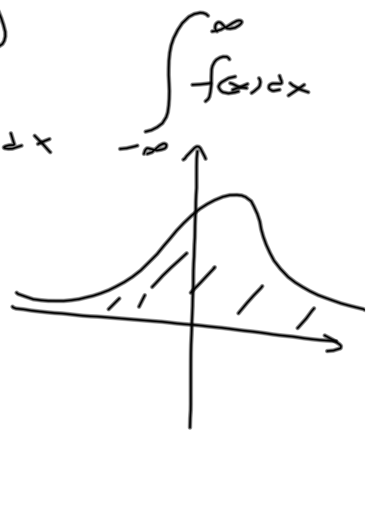
- proper integral or definite integral

  $$\int_a^b f(x)\,dx$$ where f is continuous over $[a,b]$ and a, b are constant

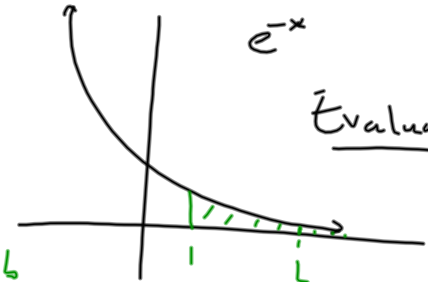
- Improper integral: $\int_{-\infty}^{b} f(x)\,dx$, $\int_a^{\infty} f(x)\,dx$, $\int_{-\infty}^{\infty} f(x)\,dx$ or $\int_a^b f(x)\,dx$ where f is not continuous over $[a,b]$







Ex:



Evaluate: $\int_1^\infty e^{-x}\,dx$

(unbounded region)

$$\int_{1=a}^{\infty=b} e^{-x}\,dx = \lim_{b\to\infty}\int_1^{b} e^{-x}\,dx$$

$$= \lim_{b\to\infty}\left[-e^{-x}\right]_1^b = \lim_{b\to\infty}\left[-e^{-b}-(-e^{-1})\right]$$

$$= \lim_{b\to\infty}\left(-\underbrace{e^{-b}}_{\to 0}\right) + e^{-1}$$

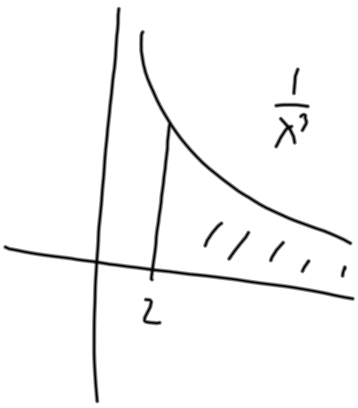
$$= e^{-1} = \frac{1}{e}$$

So $\boxed{\int_1^\infty e^{-x}\,dx = \frac{1}{e}}$

If the limit of the integral exists, we say that the integral converges to the value of the limit

$\Longrightarrow$ the area under the curve is finite

Ex: Determine whether or not the integral $\displaystyle\int_2^\infty \frac{dx}{x^3}$ converges



Solution:

$$\lim_{b\to\infty}\int_2^b \frac{1}{x^3}dx = \lim_{b\to\infty}\int_2^b x^{-3}dx$$

$$= \lim_{b\to\infty}\left[-\frac{1}{2x^2}\right]_2^b = \lim_{b\to\infty}\left[\frac{-1}{2b^2}\xrightarrow{} 0 - \left(\frac{-1}{8}\right)\right]$$

So $\displaystyle\int_2^\infty \frac{1}{x^3}dx$ Converges and $= +\frac{1}{8}$

the area is finite!

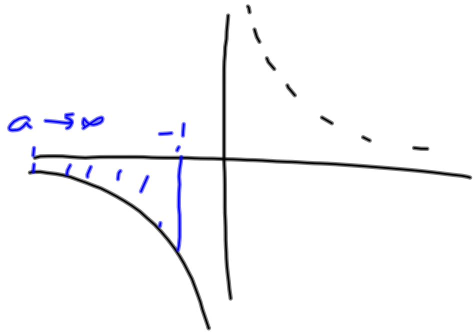
Note: $\displaystyle\int_0^2 \frac{1}{x^3}dx$ is improper!

$$= \lim_{a\to 0^+}\int_a^2 \frac{1}{x^3}dx = \lim_{a\to 0^+}\left[-\frac{1}{2x^2}\right]_a^2 = -\frac{1}{8} - \lim_{a\to 0^+}\left[\frac{-1}{2a^2}\right]$$

$= -\infty$ (DNE)

Integral diverges - The area is infinite!

Ex: $\displaystyle\int_{-\infty}^{-1} \frac{1}{x}\,dx$



$$= \lim_{a\to-\infty} \int_{a}^{-1} x^{-1}\,dx$$

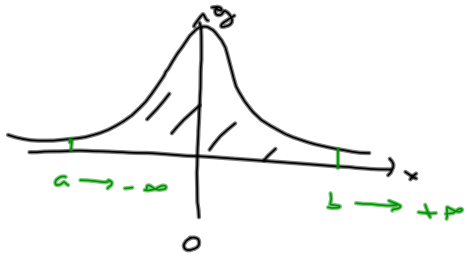
$$= \lim_{a\to-\infty} \Big[\ln|x|\Big]_{a}^{-1} = \lim_{a\to-\infty}\Big[\overset{\nearrow 0}{\ln(1)} - \ln|a|\Big]$$

$$= \lim_{a\to-\infty}\Big[-\ln|a|\Big]$$

$$= \infty \quad (\text{DNE})$$

So $\displaystyle\int_{-\infty}^{1} \frac{1}{x}\,dx$ diverges!

Ex:
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$



$$= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to-\infty} \Big[\arctan x\Big]_a^0 + \lim_{b\to\infty} \Big[\arctan x\Big]_0^b$$

$$= \arctan(0) - \lim_{a\to-\infty} \big[\arctan(a)\big] + \lim_{b\to\infty} \arctan b - \arctan(0)$$

$$= 0 - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - 0 = \boxed{\pi}$$

So the integral converges!

Show that $\displaystyle\int_a^\infty \frac{1}{x^p}\,dx = \begin{cases} \dfrac{a^{1-p}}{p-1}, & \text{if } p>1 \\ \text{diverges} & \text{if } p\le 1 \end{cases}$

with $a>1$, p any real number

<u>Case 1</u> $p=1$

$$\int_a^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\Big[\ln|x|\Big]_a^b$$

$$= \lim_{b\to\infty} \ln|b| - \ln a \qquad a>1$$

$$= \infty \qquad \text{So diverges when } p=1$$

<u>Case 2</u>: $p\neq 1$

$$\int_a^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_a^b x^{-p}\,dx \qquad (p\neq 1)$$

$$\lim_{b\to\infty}\left[\frac{1}{1-p}x^{1-p}\right]_a^b = \lim_{b\to\infty}\left[\frac{1}{1-p}b^{1-p}\right] - \left(\frac{1}{1-p}a^{1-p}\right)$$

- If $p>1$, $\displaystyle\lim_{b\to\infty}\left[\frac{1}{1-p}\cdot\frac{1}{b^{p-1}}\right] \to 0$

So $\displaystyle\lim_{b\to\infty}\int_a^\infty \frac{1}{x^p}\,dx = \frac{a^{1-p}}{p-1}$ when $p>1$

- If $p<1$, $\displaystyle\lim_{b\to\infty}\left[\frac{1}{1-p}\cdot b^{1-p}\right] \to \infty$

So $\displaystyle\lim_{b\to\infty}\int_a^\infty \frac{1}{x^p}\,dx = \infty$ (when $p<1$)

diverges!